Statistics 5314: Homework 3

For each homework assignment, turn in at the beginning of class on the indicated due date. Late assignments will only be accepted with special permission. Write each problem up *very* neatly ($\LaTeX$ is preferred). Show all of your work.

Problem 1

Show that the number of bootstrap samples, from and N vector, is:

$$\binom{2N-1}{N}.$$

Problem 2

This problem will take you through the particle filter. Consider the state and observation equations:

$$\begin{aligned} x_t &= \phi x_{t-1} + e_t \\ y_t &= x_t + \epsilon_t. \end{aligned}$$

Let $e_t \sim N(0, \sigma = 0.2)$ and $\epsilon_t \sim N(0, \sigma = 0.5)$. For our purposes, let x_t and y_t be restricted to the interval $\mathcal{A} = [-10, 10]$. Let the prior on x_0 be Uniform over $\mathcal{A}$.

Part a

For $\phi < 1$, find:

- $var(x_t)$, the marginal variance of the signal,
- $cov(x_t, x_{t\pm k})$ (covariance at lag step k),
- $E(x_t)$ and $E(y_t)$, the marginal means.
- $\pi(x_t|y_1, \ldots, y_t)$, the conditional distribution of the signal.

Part b

Simulate this process and use the particle filtering algorithm (at each time step (t)) to sample $x_t \sim \pi(x|y_1, \ldots, y_t)$. You should show plots of the actual densities at times $t = \{1, 15, 20, 25, 30\}$, along with plots of the particle filtering distribution at the same times. For each reported time, plots should show both exact and particle filtering distributions on the same plot. You may assume that this is a bounded process between [-100,100] with uniform prior distribution.

Recall that the particle filtering algorithm consists of the various steps:

1. sample $\{x_0^{(1)}, \ldots, x_0^{(N)} \sim \pi(x_0)\}$ (for the initial time step).
2. evolve the particles $x_t^{(i)} \sim q_t(x_t|x_{t-1}^i)$, for $i = \{1, \ldots, N\}$
3. weight the particles $w_t^{(i)} \sim f_t(y_t|x_t^{(i)})$, for $i = \{1, \ldots, N\}$
4. Resample from the particles $x_t^{(1)}, \ldots, x_t^{(N)}$ with weights proportional to $w^{(i)}$. These new particles represent an approximation to $\pi(x_t|y_1, \ldots, y_t)$.
5. Set t=t+1 and repeat step 2.