Statistics 5314: Homework 2

For each homework assignment, turn in at the beginning of class on the indicated due date. Late assignments will only be accepted with special permission. Write each problem up *very* neatly (LaTeX is preferred). Show all of your work.

Problem 1

The Box Mueller Algorithm follows as:

1. Sample $(u_1, u_2) \sim Unif(0, 1)$
2. Write $R = \sqrt{-2\log(u_1)}$, $\theta = 2\pi u_2$ and compute:

$$\begin{aligned} x_1 &= R\sin(\theta) \\ x_2 &= R\cos(\theta) \end{aligned}$$

.

It follows that (x_1, x_2) are iid draws from a standard normal distribution.

Part a

Implement this algorithm.

Problem 2

Consider the normal sampling density $p(x|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$.

Part a

Recall that if $x \sim t_\nu(0, 1)$ (a t-distribution with ν degrees of freedom and scale= 1 and shift= 1), then

$$g(x) \propto (1 + x^2/\nu)^{-\frac{\nu+1}{2}}.$$

Part c

Under $\nu = 1$, find the smallest M such that $p(x)/g(x) \leq M < \infty$.

Part d

Implement the following algorithm with $(\mu = 0, \sigma = 1)$

1. Sample $u \sim Unif(0, 1)$ and sample x* with density $g(x)$.
2. If $u < \frac{p(x^\star)}{Mg(x^\star)}$, set **x**=x* and exit: Else repeat step 1.

The quantity **x** has density p(x).

Part e

Plot a QQ plot of 1,000 samples given from Problem 1 against those from given from Problem 2.

Problem 3

Consider three circles each with radius 2. Let the centers of the circles have coordinates (x,y): (1/2, 0), (-1/2, 0), and (0, -1/2), respectively. Define the region ***A*** as the region which is common to all three circles.

Part a

Consider the sampling density:

$$p(x, y) \propto e^{-\frac{x^2}{2\times 0.5^2}} \times e^{-\frac{y^2}{2\times 0.5^2}} \mathbf{1}_A(x, y),$$

where $\mathbf{1}_A(x, y) = 1$ iff $(x, y) \in A$.
Sample 1,000 draws from $p(x, y)$ and plot the samples.

Problem 4

Part a

Use inverse CDF sampling to sample $\{x_1, \ldots, x_{1000}\}$ with density $\lambda \exp(-\lambda x)$. Using $\lambda = 2$, plot a histogram of your results.

Part b

Consider a uniform discretization of the standard normal density on the grid [-10,10], with resolution 0.5. Perform inverse CDF sampling on the discretized grid and plot a histogram of the results. Compare your results to those found in Problem 1.